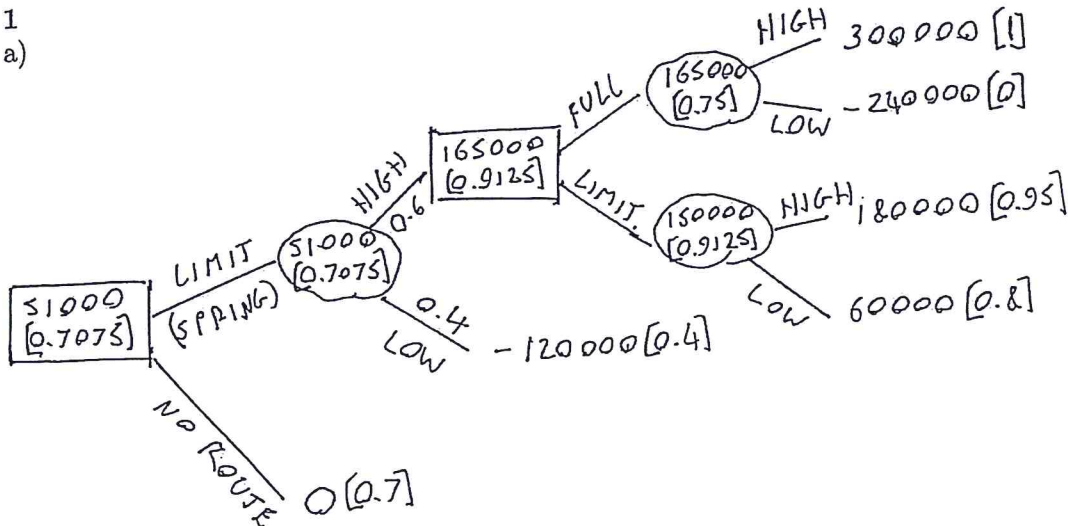


Optimization and Decision Making 2019 Examination: Solutions

1
a)



b) If they run the limited Spring service and there is high demand, then either:
they run the full service in the Summer, gaining

$$0.75 \times 300000 + 0.25 \times -240000 = 165000;$$

they run the limited service in the Summer and gain

$$0.75 \times 180000 + 0.25 \times 60000 = 150000.$$

Thus they should run the full service. This then leads to an expected reward for running the limited Spring service of $0.6 \times 165000 + 0.4 \times -120000 = 51000$. This is greater than the reward of not running the service (0), so they should run the Spring limited service and then if demand is high, run the full Summer service.

c) Letting this probability be p the expected profit for investing is $165000p - 120000(1 - p) = -120000 + 285000p$. Investment should not be made if this is less than 0, i.e. if $p < 8/19 = 0.421$.

d) We simply replace the monetary returns by their utilities in the above calculations. Thus after high demand following the limited Spring service, the expected utility for the full service is $1 \times 0.75 + 0 \times 0.25 = 0.75$ and for the limited service is $0.95 \times 0.75 + 0.8 \times 0.25 = 0.9125$. Thus the limited Summer service is now the best policy. If the limited Spring service is run the expected utility is $0.6 \times 0.9125 + 0.4 \times 0.4 = 0.7075 > 0.7$ which is the utility of 0. Thus the company should run the limited Spring service, and go for the limited Summer service if high demand occurs in the Spring.

2

a) Minimise $160x_{11} + 150x_{12} + 190x_{13} + 150x_{21} + 140x_{22} + 220x_{23} + 100x_{31} + 170x_{32} + 180x_{33} + 120x_{41} + 210x_{42} + 210x_{43}$

subject to

$$x_{11} + x_{12} + x_{13} \leq 4,$$

$$x_{21} + x_{22} + x_{23} \leq 3,$$

$$x_{31} + x_{32} + x_{33} \leq 7,$$

$$x_{41} + x_{42} + x_{43} \leq 6,$$

$$x_{11} + x_{21} + x_{31} + x_{41} \geq 5,$$

$$x_{12} + x_{22} + x_{32} + x_{42} \geq 5,$$

$$x_{13} + x_{23} + x_{33} + x_{43} \geq 10,$$

$$x_{ij} \geq 0 \text{ for } i = 1, \dots, 4; j = 1, \dots, 3.$$

b)

	1	2	3	
A	160	150	190	4
B	150	140	220	3
C	100	170	180	7
D	120	210	210	6
	5	5	10	

Using the least cost algorithm we get the following allocation

	1	2	3	
A	160	150 2	190 2	4
B	150	140 3	220	3
C	100 5	170	180 2	7
D	120	210	210 6	6
	5	5	10	

Setting $u_1 = 0$ we get $v_2 = 150, v_3 = 190, u_4 = -20, u_3 = 10, v_1 = 110, u_2 = 10$. This is not optimal since $v_1 - u_4 = 130 > 120$. Thus we add to this square (5 is the maximum), and adjust the three included squares that form a square with it, to give the following allocation

	1	2	3	
A	160	150 2	190 2	4
B	150	140 3	220	3
C	100	170	180 7	7
D	120 5	210	210 1	6
	5	5	10	

We now have $u_1 = 0, v_2 = 150, v_3 = 190, u_4 = -20, u_3 = 10, v_1 = 110, u_2 = 10$, which comparing to the vacant squares gives an optimal allocation, with cost 3170.

c) Making this change makes the solution no longer optimal, so that we need to make a further change to the allocation. One updating is sufficient, with the new optimal allocation simply being the original one from the least cost allocation, i.e.

	1	2	3	
A	160	150	190	4
		2	2	
B	150	140	220	3
		3		
C	100	170	180	7
	5		2	
D	140	210	210	6
		6		
	5	5	10	

The total cost now is 3220.

3

a) [12 marks] The dual problem is to minimise $12y_1 + 9y_2$ subject to

$$y_1 + y_2 \geq 1,$$

$$y_1 + 3y_2 \geq 2,$$

$$4y_1 + y_2 \geq 3,$$

$$y_1, y_2 \geq 0.$$

The profit line hits the feasible region at the lowest point when $y_1 = 7/11$ and $y_2 = 5/11$.

This yields a cost of $129/11$.

b) [10 marks] We use the *Equilibrium Theorem*, which is as follows:

Let $\mathbf{x}^*$ and $\mathbf{y}^*$ be feasible vectors for the standard maximum problem and its dual. Then $\mathbf{x}^*$ and $\mathbf{y}^*$ are optimal if and only if

$$y_i^* = 0 \text{ for all } i \text{ such that } \sum_{j=1}^n a_{ij}x_j^* < b_i \text{ and}$$

$$x_j^* = 0 \text{ for all } j \text{ such that } \sum_{i=1}^m y_i^* a_{ij} > c_j.$$

Using the Equilibrium Theorem we know that $x_1 = 0$ since the associated inequality for this variable was not satisfied with equality. Substituting $x_1 = 0$ into the original conditions and setting to equality gives $x_2 + 4x_3 = 12$ and $3x_2 + x_3 = 9$, which yield $x_2 = 24/11, x_3 = 27/11$. Thus we have a profit which is $129/11$, which is consistent with a).

c) [8 marks] The dual problem is to minimise $2y_1 + 9y_2$ subject to

$$y_1 + y_2 \geq 1,$$

$$y_1 + 3y_2 \geq 2,$$

$$\begin{aligned} 4y_1 + y_2 &\geq 3, \\ y_1, y_2 &\geq 0. \end{aligned}$$

Now the profit line hits the feasible region at $y_1 = 2, y_2 = 0$, with a cost of 4. Here the first and third conditions are not satisfied with equality, so that $x_1 = x_3 = 0$. Similarly $y_2 = 0$ means that the second condition of the original problem is not satisfied with equality. Substituting $x_1 = x_3 = 0$ into the first gives $x_2 = 2$, yielding a profit of 4 as expected.

4

a) [8 marks] The expected rate of return is given by
 $\mu_P = E[\sum_{i=1}^n w_i \rho_i] = \sum_{i=1}^n w_i E[\rho_i] = \sum_{i=1}^n w_i \mu_i.$

The variance is $\sigma_P^2 = E[(\rho_P - \mu_P)^2] = E[(\sum_{i=1}^n w_i \rho_i - \sum_{i=1}^n w_i \mu_i)^2] =$
 $E[\sum_{i=1}^n (w_i (\rho_i - \mu_i))^2] = \sum_{i,j=1}^n w_i w_j E[(\rho_i - \mu_i)(\rho_j - \mu_j)] = \sum_{i,j=1}^n w_i w_j \sigma_{ij}.$

b) [10 marks] We wish to minimise $f(w_1, \dots, w_n) = \frac{1}{2} \sum_{i,j=1}^n w_i w_j \sigma_{ij}$, subject to
 $g_1(w_1, \dots, w_n) = \sum_{i=1}^n w_i \mu_i - \mu_P = 0$, and $g_2(w_1, \dots, w_n) = \sum_{i=1}^n w_i - 1 = 0$.

We consider the following Lagrangian:

$L(w_1, \dots, w_n | \lambda_1, \lambda_2) = f(w_1, \dots, w_n) - \lambda_1 g_1(w_1, \dots, w_n) - \lambda_2 g_2(w_1, \dots, w_n)$. To find our optimal portfolio, we need to take all of the derivatives of the Lagrangian and set each of them to be equal to 0.

$$\frac{\partial L}{\partial w_i} = w_i \sigma_i^2 + \sum_{j \neq i}^n w_j \sigma_{ij} - \lambda_1 \mu_i - \lambda_2 = 0, \quad i = 1, \dots, n$$

$$\frac{\partial L}{\partial \lambda_1} = \mu_P - \sum_{i=1}^n w_i \mu_i = 0,$$

$$\frac{\partial L}{\partial \lambda_2} = 1 - \sum_{i=1}^n w_i = 0.$$

For three assets we must solve the simultaneous equations

$$\frac{\partial L}{\partial w_1} = w_1 \sigma_1^2 + w_2 \sigma_{12} + w_3 \sigma_{13} - \lambda_1 \mu_1 - \lambda_2 = 0,$$

$$\frac{\partial L}{\partial w_2} = w_2 \sigma_2^2 + w_1 \sigma_{12} + w_3 \sigma_{23} - \lambda_1 \mu_2 - \lambda_2 = 0,$$

$$\frac{\partial L}{\partial w_3} = w_3 \sigma_3^2 + w_1 \sigma_{13} + w_2 \sigma_{23} - \lambda_1 \mu_3 - \lambda_2 = 0,$$

$$\begin{aligned}\frac{\partial L}{\partial \lambda_1} &= \mu_P - w_1\mu_1 - w_2\mu_2 - w_3\mu_3 = 0, \\ \frac{\partial L}{\partial \lambda_2} &= 1 - w_1 - w_2 - w_3 = 0.\end{aligned}$$

After working (the key steps of which must be given) these rearrange to give

$$\begin{aligned}\mu_P - w_1\mu_1 - w_2\mu_2 - w_3\mu_3 &= 0, \\ 1 - w_1 - w_2 - w_3 &= 0, \\ [\mu_2(\sigma_1^2 - \sigma_{13}) + \mu_1(\sigma_{13} - \sigma_{12}) + \mu_3(\sigma_{12} - \sigma_1^2)]w_1 + \\ [\mu_3(\sigma_2^2 - \sigma_{12}) + \mu_2(\sigma_{12} - \sigma_{23}) + \mu_1(\sigma_{23} - \sigma_2^2)]w_2 + \\ [\mu_1(\sigma_3^2 - \sigma_{23}) + \mu_3(\sigma_{23} - \sigma_{13}) + \mu_2(\sigma_{13} - \sigma_3^2)]w_3 &= 0.\end{aligned}$$

The first two equations give

$$\begin{aligned}w_2 &= \frac{(\mu_P - \mu_3) - w_1(\mu_1 - \mu_3)}{\mu_2 - \mu_3}, \\ w_3 &= \frac{(\mu_2 - \mu_P) + w_1(\mu_1 - \mu_2)}{\mu_2 - \mu_3}.\end{aligned}$$

c) [8 marks] Substituting, we have $w_2 = (-0.03 + w_1 0.04)/(-0.03) = 1 - 4w_1/3$ and $w_3 = (0 - 0.01w_1)/(-0.03) = w_1/3$.

Using the third equation, we have

$$-0.001w_1 + 0.004w_2 + 0.002w_3 = 0 \Rightarrow -w_1 + 4w_2 + 2w_3 = 0 \Rightarrow$$

$$-w_1 + 4(1 - 4w_1/3) + 2w_1/3 = 0 \Rightarrow 17w_1 = 12 \Rightarrow w_1 = 12/17. \text{ This in turn yields } w_2 = 1/17, w_3 = 4/17.$$

d) [4 marks] Substituting, we have $w_2 = (-0.02 + w_1 0.04)/(-0.03) = (2 - 4w_1)/3$ and

$$w_3 = (-0.01 - 0.01w_1)/(-0.03) = (1 + w_1)/3.$$

The third equation is unchanged, so we have

$$-w_1 + 4(2 - 4w_1)/3 + 2(1 + w_1)/3 = 0 \Rightarrow 17w_1 = 10 \Rightarrow w_1 = 10/17. \text{ This in turn yields } w_2 = -2/17, w_3 = 9/17.$$

This would involve some short selling of the second stock. If this was not possible, we would set $w_2 = 0$ and this would give $w_1 = w_3 = 1/2$ as the solution which yields the target return.